Pink Kangaroo

a member of the Association Kangourou sans Frontières

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	B	E	A	C	B	B	B	C	B	D	C	B	E	E	C	D	C	B	D	A	B	C	A	B

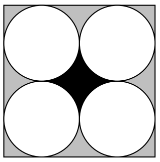
1. What is the value of $\frac{2 \times 0.24}{20 \times 2.4}$?

A 0.01 B 0.1 C 1 D 10 E 100

Solution **A**

$\frac{2 \times 0.24}{20 \times 2.4} = \frac{0.48}{48}$ which is 0.01.

2. The figure shows a square with four circles of equal area, each touching two sides of the square and two other circles. What is the ratio between the area of the black region and the total area of the grey regions?



A 1 : 4 B 1 : 3 C 2 : 3 D 3 : 4 E π : 1

Solution **B**

The figure consists of 4 white circles, each of which is inscribed in a square which is one quarter of the large square. In each of those quarters, there is one black part and three grey parts, all of equal area. So the answer is 1 : 3.

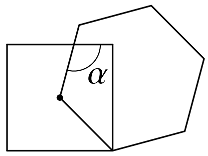
3. 232 and 111 are both three-digit palindromes as they read the same from left to right as they do right to left. What is the sum of the digits of the largest three-digit palindrome that is also a multiple of 6?

A 16 B 18 C 20 D 21 E 24

Solution **E**

For a number to be a multiple of 6, it must be even. Therefore to make the number as big as possible it must start with 8 to also end in 8. Then it must also be a multiple of 3, so its digit sum is also a multiple of 3. This leaves 828, 858 and the largest possibility, which is 888. This has a digit sum of 8 + 8 + 8, which is 24.

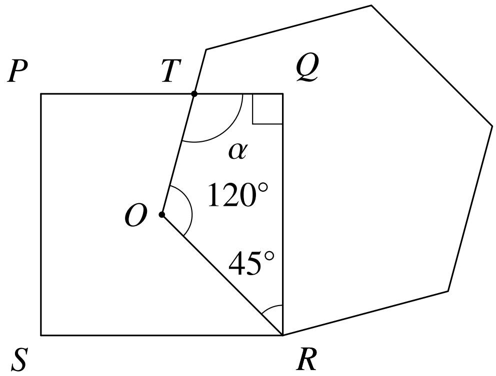
4. Tom draws a square. He adds a regular hexagon, one side of which joins the centre of the square to one of the vertices of the square, as shown. What is the size of angle α?



A 105° B 110° C 115° D 120° E 125°

Solution **A**

Let T be the vertex of the requested angle. Look at the angles of the quadrilateral $ORQT$.



$\angle ORQ = 45°$ as it is the angle between the side and diagonal of a square.

$\angle RQT$ is a right angle as it is a corner of a square.

$\angle TOR = 120°$ as it is the interior angle of a hexagon.

Angles in a quadrilateral must add to 360° hence:

$45° + 90° + 120° + \alpha = 360$ which gives $\alpha°$ as 105°.

5. Sadinie is asked to create a rectangular enclosure using 40 m of fencing so that the side-lengths, in metres, of the enclosure are all prime numbers. What is the maximum possible area of the enclosure?

A $51\,\text{m}^2$ B $84\,\text{m}^2$ C $91\,\text{m}^2$ D $96\,\text{m}^2$ E $99\,\text{m}^2$

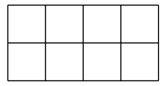
Solution **C**

Let the length and width, in metres, of the rectangle be l and w, respectively. We need consider only cases where $l < w$. Since the side length of the fence is 40 m, $40 = 2 \times (l + w)$. So $l + w = 20$.

Both l and w are prime numbers. There are 2 possible pairs $(l, w) = (3, 17), (7, 13)$.

Maximum possible area is $(7 \times 13)\ \text{m}^2$ which is $91\ \text{m}^2$

6. Lil writes one of the letters P, Q, R, S in each cell of a 2×4 table. She does this in such a way that, in each row and in each 2×2 square, all four letters appear. In how many ways can she do this?

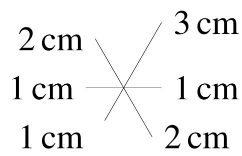


A 12 B 24 C 48 D 96 E 198

SOLUTION **B**

The letters P, Q, R and S can be arranged freely in the four squares of the first row. There are four options for the first square, then three options for the second square and two for the third square. That is $4 \times 3 \times 2 \times 1$, which is equal to 24 options.
Having filled the first row, consider the second square of the second row. This can't be the same letter as in the first, second or third position of the first row. So it must be the same letter as in the fourth position. There is now only one choice for the letters in the first and third positions of the second row and, once these are chosen, only one choice for the fourth position in that row. Therefore the second row is uniquely defined by the first row and there are 24 options.

7. The diagram gives the lengths of six lines meeting at a point. Eric wishes to redraw this diagram without lifting his pencil from the paper. He can choose to start his drawing anywhere. What is the shortest distance he can draw to reproduce the figure?



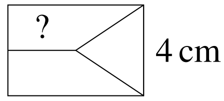
A 14 cm B 15 cm C 16 cm D 17 cm E 18 cm

SOLUTION **B**

The shortest total distance is found by starting and finishing on the longest lines. Every other line must be navigated twice, out from the centre and then back.

So the minimum length, in cm, will be found by $3 + 2 + 2 \times (2 + 1 + 1 + 1)$, which is 15 cm.

8. A rectangle is divided into three regions of equal area. One of the regions is an equilateral triangle with side-length 4 cm; and the other two are trapezia, as shown.



What is the length of the smaller of the parallel sides of the trapezia?

A $\sqrt{2}$ cm B $\sqrt{3}$ cm C $2\sqrt{2}$ cm D 3 cm E $2\sqrt{3}$ cm

Solution **B**

The height of the triangle is $2\sqrt{3}$ cm, which you can show using Pythagoras. This gives its area, in cm^2 $\frac{1}{2} \times 4 \times 2\sqrt{3}$, which is $4\sqrt{3}$ cm^2. The area of the rectangle is then $3 \times 4\sqrt{3}$ cm^2 or $12\sqrt{3}$ cm^2. Then its length is $\frac{12\sqrt{3}}{4}$ cm $= 3\sqrt{3}$ cm. So the length of the smallest parallel side of the trapezium is $3\sqrt{3} - 2\sqrt{3}$ cm,which is $\sqrt{3}$ cm.

9. The ages of Jo, her daughter and her grandson are all even numbers. The product of their three ages is 2024. How old is Jo?

A 42 B 44 C 46 D 48 E 50

Solution **C**

Let Jo have the age of $2j$, her daughter's age be $2d$ and her grandson's age be $2g$. The product of their ages is $(2g) \times (2d) \times (2j)$. We compare this with the prime factorisation of $2024 = 2 \times (2 \times 11) \times (2 \times 23)$.
So Jo is 46.

10. The sum of the digits of the positive integer N is three times the sum of the digits of $N + 1$. What is the smallest possible sum of the digits of N?

A 9 B 12 C 15 D 18 E 27

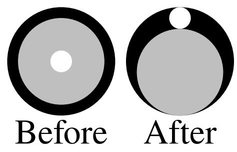
Solution **B**

The sum of the digits of a number decreases only if the number has the last digit 9, so that the next number ends in 0, and 1 is added to its penultimate digit. In that case, the sum of the digits of the number $N + 1$ decreases by 8, compared to that of the number N. At the same time, it is one third of the sum of the digits of N. Denoting the sum of the digits of N by x, we have $x = 3(x - 8)$, which gives $x = 12$.

Note that if the number ends in more that one 9, its sum of digits will be at least 19, which is already greater than 12.

The number 39 has the property described above and has a sum of digits of 12.

11. Polly has three circles cut from three pieces of coloured card. She originally places them on top of each other as shown. In this configuration the area of the visible black region is seven times the area of the white circle.



Polly moves the circles to a new position, as shown, with each pair of circles touching each other. What is the ratio between the areas of the visible black regions before and after?

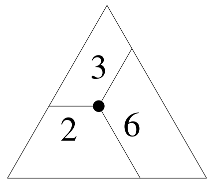
A 3 : 1 B 4 : 3 C 6 : 5 D 7 : 6 E 9 : 7

Solution **D**

The areas of the circles stay the same in the movement, therefore the amount of black covered by the grey circle is the same.

The white circle goes from being on top of the grey to covering part of the black. This will reduce the visible area of the black circle by the area of the white circle. Therefore the area is reduced by one-seventh. Hence the visible area of the black circle becomes six-sevenths of its original value. So the required ratio is $1 : \frac{6}{7} = 7 : 6$.

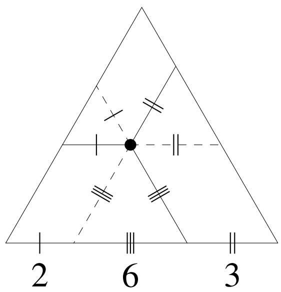
12. A point is chosen inside an equilateral triangle. From this point we draw three segments parallel to the sides, as shown. The lengths of the segments are 2 m, 3 m and 6 m. What is the perimeter of the triangle?



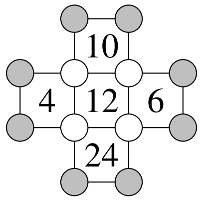
A 22 m B 27 m C 33 m D 39 m E 44 m

Solution **C**

Extend the segments through the point to the border of the shape. On the base we have an equilateral triangle and two parallelograms. Hence, the sides of the original triangle have length 2 m + 3 m + 6 m = 11 m. So the perimeter of the triangle is 33 m.



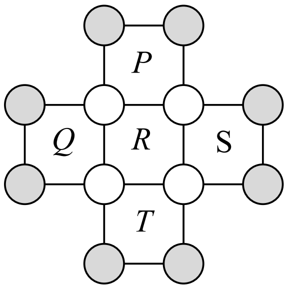
13. A number is written in each of the twelve circles shown. The number inside each square indicates the product of the numbers at its four vertices. What is the product of the numbers in the eight grey circles?

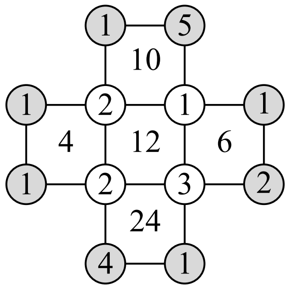


A 20 B 40 C 80 D 120 E 480

Solution **B**

If the products inside the squares are P, Q, R, S, T, then the required product is $\frac{PQST}{R^2}$. This is because the numbers in grey circles are included once each in the products P, Q, S and T and those in white circles show up twice each. So the product of the numbers in the grey circles is $\frac{10\times4\times6\times24}{12\times12} = 10 \times 2 \times 2 = 40$. The picture on the right shows an example with a possible construction.





14. The sides, in cm, of two squares are integers. The difference between the areas of the two squares is 19 c m 1. What is the sum of the perimeters of the two squares?

A 38 cm B 60 cm C 64 cm D 72 cm E 76 cm

Solution **E**

Let the side-length of the two squares, in cm, be x and y. Because the difference between their areas is $19\,\text{cm}^2$, $x^2 - y^2 = 19$. That is, $(x + y)(x - y) = 19$. Since $x + y$ and $x - y$ are positive integers, it follows that $x + y = 19$.

It follows that the sum of the perimeters is $4x + 4y$ cm, which is $4(x + y)$ cm $= (4 \times 19)$ cm, which is 76 cm.

15. Molly has a set of cards numbered 1 to 12. She places eight of them at the vertices of an octagon so that the sum of every pair of numbers at opposite ends of an edge of the octagon is a multiple of 3.

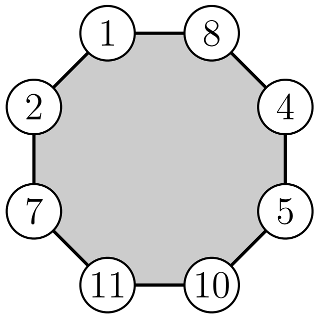
Which numbers did Molly not place?

A 1, 5, 9 and 12 B 3, 5, 7 and 9 C 1, 2, 11 and 12
D 5, 6, 7 and 8 E 3, 6, 9 and 12

Solution **E**

If one of the numbers at a vertex is a multiple of 3 then the number at *every* neighbouring vertex must also be a multiple of 3.

Since we have only 4 multiples of 3 we come to a contradiction. So the numbers 3, 6, 9, 12 must be removed. The diagram on the right shows one way the remaining numbers could be placed.



16. Peter always tells the truth or always lies on alternate days. One day, he made exactly four of the following five statements. Which one did he not make?

A I lied yesterday and I will lie tomorrow.
B I am telling the truth today and I will tell the truth tomorrow.
C 2024 is divisible by 11.
D Yesterday was Wednesday.
E Tomorrow will be Saturday.

Solution **C**

Statement (B) can only be made by a person who lies on that day. Looking at the statements (D) and (E), they cannot both be true. Therefore, we have at least two statements that are lies. Therefore Peter has not made four true statements. This means that Peter is lying today. However statement (C) is true because $2024 = 11 \times 184$. So this is the statement that Peter did not make.

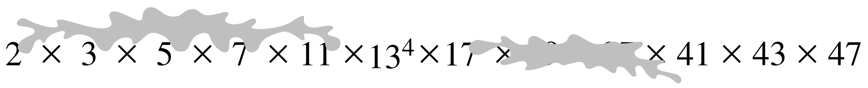
17. Matthew rolled a normal die 24 times. All numbers from 1 to 6 came up at least once. The number 1 came up more times than any other number. Matthew added up all the numbers. The total he obtained was the largest one possible. What total did he obtain?

A 83 B 84 C 89 D 90 E 100

SOLUTION **D**

Since Matthew has rolled each number at least once, we can guarantee that 6 of the rolls have a sum of $1 + 2 + 3 + 4 + 5 + 6 = 21$. Let's consider the remaining 18 rolls.
There should be more 1s than any other numbers. It is clear that the number of sixes rolled should be one less than the number of ones in order to make the sum as large as possible.
If we have 9 ones: then 8 sixes adding a five we get $9 + 48 + 5 = 62$.
If we have 8 ones: then 7 sixes adding 3 fives we get $8 + 42 + 15 = 65$.
If we have 7 ones: then 6 sixes adding 5 fives we get $7 + 36 + 25 = 68$.
If we have 6 ones: then 5 sixes, 5 fives and adding 2 fours we get $6 + 30 + 25 + 8$, which is 69.
This leads to a maximum total of 90.
Further reduction of 1s does not increase the maximum possible sum.

18. For some positive integer n, the prime factorisation of the number $n! = 1 \times 2 \times \ldots \times n$ is of the form shown below.



The primes are written in increasing order. Ink has covered some of the primes and some of the exponents. What is the exponent of 17?

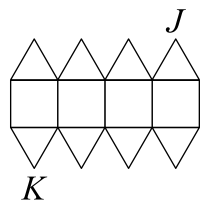
A 1 B 2 C 3 D 4 E 5

SOLUTION **C**

The appearance of the prime 47 shows that $47 \leq n < 53$ (otherwise we would see the next prime 53).

Since 13^4 is part of the factorisation, 4 multiples of 13 are included in $n!$. These are 13, 26, 39 and 52, so $n = 52$. There are three multiples of 17 before 52 (these are 17, 34 and 51). Therefore, the exponent of 17 in this factorisation is 3.

19. Kevin makes a net using a combination of squares and equilateral triangles, as shown in the figure. The side-length of each square and of each triangle is 1 cm. He folds the net up to form the surface of a polyhedron. What is the distance between the vertices J and K in this polyhedron?

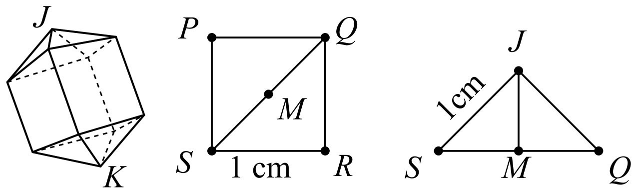


A $\sqrt{5}$ cm B $(1+\sqrt{2})$ cm C $\frac{5}{2}$ cm

D $(1+\sqrt{3})$ cm E $2\sqrt{2}$ cm

Solution **B**

The net is folded to form a polyhedron that consists of a cube with a square based pyramid on the top and bottom faces. J becomes the apex of the pyramid on the top face, and K the apex of the pyramid on the bottom face.



Therefore the distance between J and K equals the height of the cube plus the heights of the pyramids.

The 4 vertices joined to J form a square $PQRS$ of side 1 cm long. Therefore, by the Pythagorean theorem, its diagonal SQ is $\sqrt{2}$ cm long. Let M be the midpoint of SQ. Then JMQ is a right triangle with hypotenuse JQ of side 1 cm and leg QM of side $\frac{\sqrt{2}}{2}$ cm. Again, using the Pythagorean theorem, we get

$$|JM|^2 = 1 - \left(\frac{\sqrt{2}}{2}\right)^2 = 1 - \frac{2}{4} = \frac{1}{2},$$

thus

$$|JM| = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

The distance between J and K is

$$\frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} = 1 + \sqrt{2} \text{ cm}.$$

20. Jill has some unit cubes which are all black, grey, or white. She uses 27 of them to build a $3 \times 3 \times 3$ cube. She wants the surface to be exactly one-third black, one-third grey, and one-third white. The smallest possible number of black cubes she can use is X and the largest possible number of black cubes she can use is Y. What is the value of $Y - X$?

A 1 B 3 C 6 D 7 E 9

SOLUTION **D**

The surface area of the cube is $6 \times 3^2 = 54$. This means that the black, the grey and the white area should equal $18\,\text{cm}^2$ each.
Each unit cube can contribute 3 if it is a corner cube, 2 if it is an edge cube, and 1 if it is a centre of a face.
There are 8 corner cubes, 12 edge cubes and 6 centres of faces. 1 unit cube is completely hidden from the surface. The smallest number of unit cubes to make area of 18 is $X = 6$ corner cubes. The largest number of unit cubes to make area of 18 is 12: 6 centres of faces and 6 edge cubes. Indeed, we can choose the "hidden" unit cube to have the same colour, so we would have used $Y = 13$ black unit cubes. In other words, $Y = 13, X = 6, Y - X = 7$.

21. Meera walked in the park. She walked half of the total time at a speed of 2 km/h. She then walked half of the total distance at a speed of 3 km/h. Finally, she completed the remainder of the walk at a speed of 3 km/h. For what fraction of the total time did she walk at a speed of 4 km/h?

A $\frac{1}{14}$ B $\frac{1}{12}$ C $\frac{1}{7}$ D $\frac{1}{5}$ E $\frac{1}{4}$

SOLUTION **A**

Let's denote the half of the time for Meera's walk with t and the time she walks with a speed of 4 km/h with x. Then, the time she walks with a speed of 3 km/h is $t - x$. We have the following:
Speed: 2 km/h, Time: t, Distance : $2t$
Speed: 3 km/h, Time: $t - x$, Distance : $3(t - x)$
Speed: 4 km/h, Time: x, Distance : $4x$
Because Meera walks half the distance at 3 km/h, we have $3(t - x) = 2t + 4x$. Therefore $t = 7x$. Hence $x = \frac{1}{7}t$. The total time for the walk was $2t$, so $x = \frac{1}{14} \times 2t$. That is, $\frac{1}{14}$ of the time was spent walking with a speed of 4 km/h.

22. Given the integers from 1 to 25, Ajibola wants to remove a few and then split those that remain into two groups so that the products of the integers in each group are equal. Ajibola removes the smallest possible number of integers in order to achieve this. What is the sum of the numbers which Ajibola removes?

A 75 B 79 C 81 D 83 E 89

Solution **B**

Thinking of the prime factorisation of the products of the numbers of both groups, they have to be the same, since the products are equal. Therefore, prime numbers that have only one multiple among the numbers from 1 to 25 have to be removed. These prime numbers are 13, 17, 19 and 23. 7 has three multiples among the numbers from 1 to 25: 7, 14 and 21. It is left to the reader to show that all the primes other than 7, 13, 17, 19 and 23, that occur in the prime factorisation occur to an even power. So the smallest number of primes which need to be removed is five and these are 7, 13, 17, 19 and 23. Their sum is 79. The remaining 20 numbers can be split in two groups that have equal products.
For example:

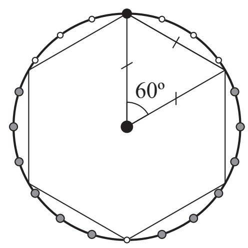
$$3\times5\times8\times14\times15\times18\times20\times22\times24 = 2\times4\times6\times9\times10\times11\times12\times16\times21\times25 = 2^{11}\times3^5\times5^3\times7\times11.$$

23. Twenty points are equally spaced around the circumference of a circle. Kevin draws all the possible chords that connect pairs of these points. How many of these chords are longer than the radius of the circle but shorter than its diameter?

A 90 B 100 C 120 D 140 E 160

Solution **C**

Each of the 20 points is connected with the 19 other points on the circumference, forming 19 chords. One of these 19 chords is exactly equal to the diameter, leaving 18 chords to consider.



For a chord to be larger than the radius, the angle it subtends at the centre has to be larger than 60°. This is because the chord and two radii form an equilateral triangle as shown in the diagram.

Since the 20 points form a regular 20-gon, the central angle of the chord connecting two adjacent points is $360° \div 20$ which is 18°. A chord PQ has a length greater than the radius but less than the diameter provided that it subtends an angle strictly between 60° and 180°. Since $3 \times 18° = 54° < 60°$. this means that there must be at least 3 of the 20 points between P and Q, the shortest way round, and that PQ must not be a diameter.

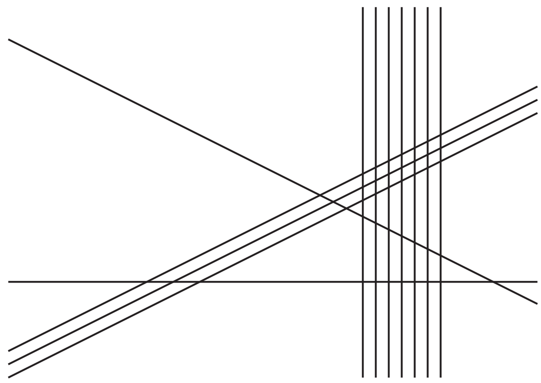
That is, three pairs of chords from a point to its three nearest points along the circumference (on either side) are going to be shorter than the radius. This leaves us with six pairs of chords, or twelve chords per point. Since each chord is counted twice (once per endpoint) the total number of chords satisfying the condition is $(20 \times 12) \div 2 = 120$.

24. There are n distinct lines in the plane. One of these lines intersects exactly 5 of the n lines, another of these intersects exactly 9 of the n lines, and yet another intersects exactly 11 of them. Which of the following is the smallest possible value of n?

A 12 B 13 C 14 D 25 E 28

SOLUTION **A**

Since one of the lines intersects 11 other lines, the minimum number of lines is at least 12. To show that exactly twelve lines can satisfy the requirement, consider a family of seven parallel lines, another family of three parallel lines not parallel to the first family and two other lines, intersecting each other and intersecting both families of parallel lines, as shown.



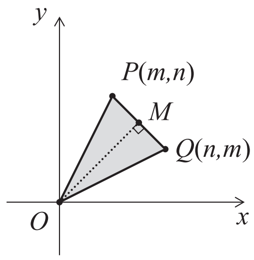
Any line in the family of seven intersects 5 other lines, any of the lines in the second family of three parallel lines intersects 9 other lines, either of two lines not in either family intersects 11 other lines.

The diagram shows a possible construction with this minimum number of lines. Therefore, the answer is 12.

25. Suppose m and n are integers with $0 < m < n$. Let $P = (m, n)$, $Q = (n, m)$, and $O = (0, 0)$. For how many pairs of m and n will the area of triangle OPQ be equal to 2024?

A 4 B 6 C 8 D 10 E 12

SOLUTION **B**



Taking PQ as the base, triangle OPQ has height $|OM|$, where M is the midpoint of PQ and is given by:

$$M = \left(\frac{n+m}{2}, \frac{n+m}{2}\right)$$

By Pythagoras,

$$|PQ| = \sqrt{(m-n)^2 + (n-m)^2} = \sqrt{2}|n-m|.$$

As $m < n$, $|PQ| = \sqrt{2}(n-m)$. The length of OM, can be found similarly:

$$|OM| = \sqrt{\left(\frac{n+m}{2}\right)^2 + \left(\frac{n+m}{2}\right)^2} = \frac{n+m}{\sqrt{2}}.$$

Therefore the area of the triangle OPQ is given by,

$$\frac{1}{2}(|PQ| \times |OM|) = \frac{1}{2}\left(\sqrt{2} \times (n-m) \times \frac{n+m}{\sqrt{2}}\right) = \frac{(n+m)(n-m)}{2}.$$

Hence $\frac{(n+m)(n-m)}{2} = 2024$, therefore $(n+m)(n-m) = 4048$.

Since m and n are both integers $n+m$ and $n-m$ are both even or both odd.

There are six ways that $4048 = 2^4 \times 11 \times 23$ can be represented as a product of two even numbers: 2×2024, $2^2 \times 1012$, $2^3 \times 506$, $(2 \times 11) \times (2^3 \times 23)$, $(2^2 \times 11) \times (2^2 \times 23)$, $(2^3 \times 11) \times (2 \times 23)$, so there are 6 possible pairs of (n, m) : $\{(1013, 1011), (508, 504), (257, 249), (103, 81), (68, 24), (67, 21)\}$.